

# Handelman's theorem for an order unit normed space

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## Abstract

We give a detailed proof D. Handelman's theorem stating (in the context of an order unit normed space) that a monotone  $\sigma$ -complete order unit normed space is a Banach space.

## 1 Handelman's theorem

In what follows,  $(V, u)$  denotes an order unit normed space. By definition,  $V$  is *monotone  $\sigma$ -complete* iff every ascending sequence in  $V$  that is bounded above has a supremum in  $V$ . Our proof of the following theorem is based on the proof of [1, Proposition 3.9].

**1.1 Theorem.** [Handelman] *If  $(V, u)$  is monotone  $\sigma$ -complete, then  $V$  is norm complete (i.e., a Banach space).*

*Proof.* Suppose that  $V$  is monotone  $\sigma$ -complete and let  $(a_n)_{n=0}^\infty$  be a Cauchy sequence in  $V$ . Then there exists  $0 < \beta \in \mathbb{R}$  such that  $-\beta u \leq a_n \leq \beta u$  for all  $n = 0, 1, 2, \dots$ , and by replacing each  $a_n$  by  $\beta^{-1}a_n$ , we can assume that  $-u \leq a_n \leq u$ , i.e.,  $\|a_n\| \leq 1$ , for  $n = 0, 1, 2, \dots$ . Also, we can and do replace  $a_0$  by 0 without affecting the hypothesis that  $(a_n)_{n=0}^\infty$  is a Cauchy sequence.

For each  $n \in \mathbb{N}$ , there exists  $M_n \in \mathbb{N}$  such that

$$M_n \leq i, j \in \mathbb{N} \Rightarrow \|a_i - a_j\| < 2^{-n},$$

and we can assume without loss of generality that

$$M_1 < M_2 < M_3 < \dots.$$

Thus, for  $n = 1, 2, 3, \dots$ ,

$$\|a_{M_{n+1}} - a_{M_n}\| < 2^{-n}.$$

It will be sufficient to prove that the subsequence of  $(a_n)_{n=0}^\infty$  given by

$$a_0, a_{M_1}, a_{M_2}, a_{M_3}, \dots$$

converges. Replacing  $(a_n)_{n=0}^\infty$  by this subsequence, we have

$$a_0 = 0 \text{ and for all } n \in \mathbb{N}, \|a_n\| \leq 1 \text{ and } \|a_{n+1} - a_n\| < 2^{-n}.$$

Thus,

$$a_0 = 0, -u \leq a_n \leq u \text{ and } -2^{-n}u < a_{n+1} - a_n < 2^{-n}u \text{ for all } n \in \mathbb{N}. \quad (1)$$

Now put

$$b_n := 2^{-n}u + a_{n+1} - a_n \text{ for } n = 0, 1, 2, 3, \dots$$

In particular,  $b_0 = u + a_1$ , and since  $-u \leq a_1 \leq u$ , we have  $0 \leq u + a_1 \leq 2u$ , whence  $0 \leq b_0 \leq 2u$ . Also, for  $n \in \mathbb{N}$ ,  $0 \leq b_n \leq 2(2^{-n})u$  and therefore

$$0 \leq b_n \leq 2(2^{-n})u \text{ for } n = 0, 1, 2, 3, \dots \quad (2)$$

Consider the partial sums

$$s_m := \sum_{n=0}^m b_n = a_m + 2u - 2^{-m}u. \quad (3)$$

By (2),  $(s_m)_{m=0}^\infty$  is monotone increasing and

$$0 \leq s_m = \sum_{n=0}^m b_n \leq \sum_{n=0}^m 2(2^{-n})u \leq 4u - 2^{1-m}u \leq 4u \text{ for } m = 0, 1, 2, 3, \dots; \quad (4)$$

therefore  $s := \bigvee_{m=0}^\infty s_m$  exists in  $V$ .

Temporarily fix  $m \in \{0, 1, 2, 3, \dots\}$ . Then

$$\text{for } m < p \in \mathbb{N}, \quad 0 \leq s_p - s_m = \sum_{k=m+1}^p b_k. \quad (5)$$

Thus  $(s_p - s_m)_{p=m+1}^\infty$  is a monotone increasing sequence in  $V$ , and by (4),

$$0 \leq s_p - s_m \leq s_p \leq 4u \text{ for } m < p \in \mathbb{N},$$

whence  $\bigvee_{p=m+1}^\infty (s_p - s_m)$  exists in  $V$  for  $m = 0, 1, 2, 3, \dots$ .

Since  $(s_n)_{n=0}^\infty$  is monotone increasing, it follows that

$$s = \bigvee_{n=0}^\infty s_n = \bigvee_{p=m+1}^\infty s_p \text{ for } m = 0, 1, 2, 3, \dots, \quad (6)$$

and by (6) and (5), we have

$$0 \leq s - s_m = \left( \bigvee_{p=m+1}^\infty s_p \right) - s_m = \bigvee_{p=m+1}^\infty (s_p - s_m) = \bigvee_{p=m+1}^\infty \left( \sum_{k=m+1}^p b_k \right). \quad (7)$$

By (2), for all  $m = 0, 1, 2, 3, \dots$  and all  $p > m$ ,

$$\sum_{k=m+1}^p b_k \leq \sum_{k=m+1}^p 2 \cdot 2^{-k}u \leq \sum_{k=m+1}^\infty 2 \cdot 2^{-k}u = 2 \cdot 2^{-m}u,$$

whence

$$\bigvee_{p=m+1}^\infty \left( \sum_{k=m+1}^p b_k \right) \leq 2 \cdot 2^{-m}u.$$

Consequently, by (7),  $0 \leq s - s_m \leq 2 \cdot 2^{-m}u$ , and therefore  $\|s - s_m\| \leq 2 \cdot 2^{-m}$  for  $m = 0, 1, 2, \dots$  and we have

$$s = \lim_{m \rightarrow \infty} s_m. \quad (8)$$

Thus by (3) and (8),

$$\lim_{m \rightarrow \infty} a_m = \lim_{m \rightarrow \infty} (s_m + 2^{-m}u - 2u) = s - 2u. \quad \square$$

## References

- [1] D. Handelman: Rings with involution as partially ordered aberlian groups, Rocky Mt.J. Math. **11** (1981) 337–381.